Clustering and synchronization with positive Lyapunov exponents

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Abstract

Clustering and correlation effects are frequently observed in chaotic systems in situations where, because of the positivity of the Lyapunov exponents, no dimension reduction is to be expected. In this paper, using a globally coupled network of Bernoulli units, one finds a general mechanism by which strong correlations and slow structures are obtained at the synchronization edge. A structure index is defined, which diverges at the transition points. Some conclusions are drawn concerning the construction of an ergodic theory of self-organization.

1 Introduction

Even simple one degree of freedom systems may display a rich dynamical behavior, namely, sensitive dependence to initial conditions (chaos), periodic and aperiodic orbits of all types, mixing properties, etc. When these systems are coupled, their cooperative behavior reveals a set of dynamical patterns of which the most interesting ones are clustering, coherent structures and synchronization. The cooperative effects of simple coupled systems and their dependence on the intensity of the coupling seem to provide the dynamical basis for many phenomena in physics[1] [2], chemistry[3] [4] and the neurosciences[5] [6] [7]. It has also been suggested by several authors[8] [9]

[10] that self-synchronized activity in insect colonies has adaptive advantages and is responsible for efficient task fulfillment.

Synchronization of chaotic systems is a very interesting phenomenon that has been extensively studied[11] [12] and which, in addition to its role in modeling natural systems, may have technological applications, for example in the field of secure communications. According to the theory developed by Pecora and Carrol a subsystem synchronizes with another (chaotic) subsystem when the corresponding conditional Lyapunov exponents are negative. The conditional exponents being bona-fide ergodic invariants[13], this is a precise mathematical condition for synchronization on the support of the invariant measure where the exponents are defined.

In the standard synchronization scenario, one of the chaotic subsystems enslaves the other and there is in an effective dimension reduction in phase-space. However it has been noticed by several authors that there are situations where one obtains synchronization even with positive conditional exponents[14] and clustering or strong correlations of the subsystems even when they are desynchronized[15] [16] [17]. These correlations have been called *hidden coherence*. Synchronization with positive conditional exponents has been attributed to the extreme trap effect[14], namely the fact that near an extreme point of the iteration function the linear terms vanish and second order terms may have an effective contracting role. On the other hand fluctuations in the mean field felt by each individual subsystem and instability of the solutions for an effective one-dimensional Perron-Frobenius equation, have been proposed[18] as an explanation for the hidden coherence. These mechanisms may of course play a role in the formation of coherent structures and specific dynamical features must surely have to be taken into account for the concrete interpretation of each particular case. However one would like to understand why clustering and correlation effects are so common in coupled situations where naively we would still expect to have very small or no dimension reduction in the overall dynamics.

The method to be used in this paper is also to study a concrete example but one that is sufficiently simple for almost everything to be exactly computed and from which essential features may be isolated from model details. In particular, by using piecewise linear maps one gets rid of second order effects and non-uniform hyperbolicity. Also the fluctuations in the mean field, seen by each element of the coupled system, seem to be rather tame and the origin of the correlations and structures may be correctly identified. Once

the behavior of these phenomena is clearly understood in this model we will then attempt to separate what seems to be universal features and what are particular features of the model.

2 Correlations at the synchronization edge

Consider a globally coupled system of N Bernoulli units with dynamics

$$x_i(t+1) = (1-c)f(x_i(t)) + \sum_{j\neq i}^{N} \frac{c}{N-1} f(x_j(t)) \tag{1}$$

and $f(x) = 2x$ (mod. 1). The nice feature of this globally coupled system is that, although each isolated unit is mixing and has orbits of all types, nevertheless almost everything in the coupled system may be exactly computed. This avoids interpretation ambiguities of the results and, hopefully, will allow for the separation of universal features from those that are model-dependent (see the conclusions). Except for $c = c_s = \frac{1}{2}\frac{N-1}{N}$ the system is uniformly hyperbolic and the Lyapunov exponents are:

$$\begin{aligned} \lambda_1 &= \log 2 \\ \lambda_i &= \log\left(2\left(1-\tfrac{N}{N-1}c\right)\right) \quad \text{with multiplicity} \quad N-1 \end{aligned} \tag{2}$$

For coupling strength $c < c_s$ the Lyapunov dimension is N and one expects to have a BRS-measure absolutely continuous with respect to the Lebesgue measure in R^N. This is indeed so, the distribution of the values taken by any one unit x_i is essentially flat and, for large N, the mean field seen by any one unit has very small fluctuations. However as one approaches $c = c_s$ from below, one sees that the dynamics organizes itself into synchronized patches, with each patch maintaining also an approximately constant phase relation with the other patches. The synchronization and phase locking effects however are not absolutely stable phenomena, the composition and phases of the clusters changing in time but at a very slow time scale. This clustering effects are evident on the statistics of the coordinate differences $|x_i - x_k|$ shown in Fig.1, where one has taken the time averages over all pairs of units for a 100-units system. Fig.1 shows these distributions as c varies. In Fig.1a, without interaction ($c = 0$) the triangular form of the distribution only reflects the projection along the diagonal of a uniform

distribution of the two coordinates on the unit square. As c increases this distribution is deformed (Fig.1b, for example) but nothing dramatic occurs until one reaches the region near $c_s = \frac{1}{2}\frac{N-1}{N}$. Then as shown in Fig.1c, well defined structures develop which correspond either to synchronization (peak at zero) or to approximate phase locking (peak near 0.5 and bump around 0.25). Notice however that this figure actually shows the superposition of two effects. If instead of taking the average over all distance pairs, one fixes a specific pair of coordinates, one may obtain for short periods either the two peaks at zero and 0.5 or the bump at 0.25. Finally for $c > c_s$ one obtains global synchronization (Fig.1d). Marked structures are also obtained for some other linear combinations of the coordinates. Fig.2a-d shows the statistics for $x_i + x_{i+1} - 2x_{i+2}$.

The interpretation of these effects follows nicely from the knowledge of the Lyapunov exponents listed in (2). For $c < c_s$ all Lyapunov exponents are positive. However, near c_s there is one large Lyapunov exponent whereas all the others are nearly zero. This implies a fast *separation dynamics* (sensitive dependence to initial conditions) in one direction and very slow separation dynamics in all other directions transversal to the fast one. The fast one corresponds to the eigenvector $(1, 1, 1, 1, ..., 1)$. Therefore, although the invariant physical measure is still absolutely continuous with respect to Lebesgue, the slow separation dynamics in the transversal directions corresponds to long wavelength effects in phase space that are most sensitive to the boundary conditions and the available phase-space. Then, the slow temporal structures beget non-uniform probability distributions in the linear combinations of the variables that correspond to the slow eigenvalues. In particular, $x_i - x_{i+1}$ corresponds to the eigenvector $(0, ..., 1, -1, 0, ..., 0)$ and $x_i + x_{i+1} - x_{i+2}$ to $(0, ..., 1, 1, -2, 0, ..., 0)$.

In conclusion: the existence of structures near the transition points where one or more Lyapunov exponents approach zero from above should be an universal phenomena, whereas the detailed form of the structures must depend one the particular nature of the available phase-space. The non-universality of the shape of the probability distribution is quite apparent in our example. BRS-measures are in general obtained from the topological pressure when

the function is the sum of the positive Lyapunov exponents, namely[19]

$$\mu_\phi(dX) = \lim_{\varepsilon\to 0}\lim_{T\to\infty}\sup_S \frac{1}{Z(\varepsilon,T,\phi)}\sum_S \exp\left(\int_{-T}^{T}\phi(f^tY)dt\right)\times\frac{1}{2T}\int_{-T}^{T}\delta(X-f^tY)dtdX \tag{3}$$

S being a $(\varepsilon, T)-$separated subset. If the function ϕ is the sum of the positive Lyapunov exponents, in our example it only contributes a phase factor, constant all over phase-space. Hence, the shape of the invariant measure depends only on the delta function, that is on the orbit structure which is determined by the boundary conditions and the available phase-space.

For the coupled Bernoulli units example the shape of the probability structures may actually be recovered from an approximate probabilistic equation. From (1) it follows that for any two units one has the following relation

$$x_i(t+1) - x_k(t+1) = (1 - \frac{N}{N-1}c)\left(f(x_i(t)) - f(x_k(t))\right) \tag{4}$$

However (4) does not define a deterministic dynamical law because it has two branches in the interval $(0, 0.5)$, namely

$$\begin{array}{lll} x_i - x_k & \rightarrow & \left(\begin{array}{c} 2(1-\frac{N}{N-1}c)(x_i - x_k) \\ \text{or} \\ (1-\frac{N}{N-1}c)\left(1-2(x_i-x_k)\right) \end{array}\right) \text{if } (x_i - x_k) < 0.5 \\ x_i - x_k & \rightarrow & (1-\frac{N}{N-1}c)\left(2(x_i-x_k)-1\right) \text{ if } (x_i - x_k) > 0.5 \end{array} \tag{5}$$

Defining $r = |x_i - x_k|$ and assigning probabilities p_1, p_2 and p_3 to these three branches one may write a probabilistic version of the Perron-Frobenius equation

$$\frac{\rho(r)}{1-r} = \sum_{y\in f^{-1}(r)} p_i \frac{1}{f\,'(y_i)}\frac{\rho(y_i)}{1-y_i} \tag{6}$$

where the sum runs over the three possible inverses of r and the factors $1-r$ and $1-y_i$ account for the projection along the diagonal on the unit square. Iteration of this equation with $p_1 = p_2 = 0.5$ and $p_3 = 1$ shows that in the neighborhood of $\frac{N}{N-1}c = 0.5$ one obtains the observed structures, namely either two peaks at zero and 0.5 or a bump around 0.25.

3 Self-organization and the structure index

In general one calls coherent structure (in a collective system) an identifiable phenomenon that has a scale very different from the scale of the components of the system. A structure in space will correspond to a feature at a length scale larger than the characteristic size of the components and a structure in time is a phenomenon with a time scale larger than the cycle time of the individual components. This suggests the definition of a (temporal) *structure index*

$$S = \frac{1}{N} \sum_{i=1}^{N_s} \frac{T_i - T}{T} \tag{7}$$

where N is the total number of components (degrees of freedom) of the coupled system, N_s is the number of structures, T_i is the characteristic time of structure i and T is the cycle time of the components (or, alternatively the characteristic time of the fastest structure). A similar definition would apply for a spatial structure index, by replacing characteristic times by characteristic lengths. In our coupled Bernoulli units example the characteristic times of the separation dynamics are the inverse of the Lyapunov exponents and one obtains

$$\begin{aligned} S &= \tfrac{N-1}{N} \left(\tfrac{\log 2}{\log 2\left(1 - \frac{N}{N-1}c\right)} - 1 \right) && \text{for } \tfrac{N}{N-1}c < 0.5 \\ S &= 0 && \text{for } \tfrac{N}{N-1}c > 0.5 \end{aligned} \tag{8}$$

For $\frac{N}{N-1}c > 0.5$ the structure index vanishes because the synchronized motion is effectively one-dimensional and the characteristic time of the synchronized motion coincides with the characteristic time of the individual units. The structure index is zero both for the uncoupled case and the fully synchronized one and diverges at the synchronization transition.

In a previous paper[13], the self-organization that occurs when identical dynamical systems are coupled was characterized by ergodic invariants constructed from the conditional exponents. Namely, a *measure of dynamical self-organization* was defined by

$$I = \sum_{k=1}^{N} \{h_k + h_{m-k} - h\} \tag{9}$$

where h_k and h_{N-k}, the *conditional exponent entropies* associated to the splitting $R^k \times R^{N-k}$, are the sums of the positive conditional exponents

$$\begin{array}{lcl} h_k & = & \sum_{\xi_i^{(k)}>0} \xi_i^{(k)} \\ h_{N-k} & = & \sum_{\xi_i^{(N-k)}>0} \xi_i^{(N-k)} \end{array} \tag{10}$$

the conditional exponents being the eigenvalues of

$$\lim_{n\to\infty} \left(D_k f^{n*}(x) D_k f^n(x) \right)^{\frac{1}{2n}}$$

where $D_k f^n$ is the $k \times k$ diagonal block of the full Jacobian.

For the coupled Bernoulli units example, for splittings into 1 and $N-1$ parts, one obtains in the limit of large N

$$\begin{array}{lcll} I & = & \frac{c^2}{1-c} & c \leq \frac{1}{2} \\ & = & -c & c \geq \frac{1}{2} \end{array} \tag{11}$$

This quantity is also peaked at the synchronization point, (although finite) but its interpretation is different from the structure index. The Lyapunov exponents or the conditional exponents measure the change in the dynamics that occurs when one makes a small change in the initial conditions. Therefore a system with large exponents has a large freedom to change its future state with a small effort at the present time. From this point of view, h_k measures the apparent (from the point of view of unit k) dynamical freedom (or rate of information production) of unit k. h_{N-k} has the same interpretation for the system composed of the remaining $N-1$ units. However it is h that defines the actual rate of information production (or dynamical freedom) of the whole system. Therefore I is a measure of the *apparent* excess of dynamical freedom.

4 Conclusions

1. When in an interacting multi-unit system the parameters changes and one reaches a region where one or more of the positive Lyapunov exponents approaches zero, the slow separation dynamics along the direction of the corresponding eigenvectors leads to the development of temporal structures, but without dimension reduction in phase-space. The structures are expected

to be metastable, but with a time scale much larger than the cycle time of the individual units. These regions, where the system displays what is perhaps its most interesting behavior, are located near the transition disorder-order but still on the disorder side. Emergence of structures at the Lyapunov exponents transition regions is expected to be an universal phenomenon, but the detailed nature of the structures must be model-dependent.

2. When there is a natural limitation on the range of values that the state variable of each individual unit can take, the coupling must be a convex coupling like in the Bernoulli units example. Then the convex coupling leads to an overall contracting effect and Lyapunov transitions are to be expected when the coupling increases. In spatially extended systems, for example, the basic interaction law might not change but a change in density implies an effective coupling increase. Therefore in a evolving system where the number of individuals changes in time (but the available space remains fixed), effects of the type described here might be expected to arise when the population density changes.

That at transition regions between chaos and order, evolving systems display interesting structured properties had been suggested before by several authors[20] [21]. Why some natural systems might have evolved to such narrow regions in parameter space is, to a large extent, an open question. The density-dependent increase of the effective interaction and the contracting effect implied by the convex coupling, when the amount of available phase-space remains constant, is a dynamical mechanism that might explain, in some cases, the evolution towards the transition regions.

3. Given an invariant measure for the interacting system, the structure index and the measure of self-organization are both well-defined ergodic invariants which characterize different aspects of the collective behavior. They provide a first step towards a rigorous *ergodic theory of self-organization*. In this connection it should be mentioned that a dynamical measure is not completely characterized by the Lyapunov and the conditional exponents. Ruelle, for example, has pointed out that the exponents being obtained as limits of averages, the moments of the fluctuations around the average are new, independent, ergodic invariants (unless the fluctuations are Gaussian). Moments are not always a reliable way to characterize stochastic processes[22] but large families of ergodic invariants may be obtained in several other ways, for example from a variational formulation of the dynamics[23].

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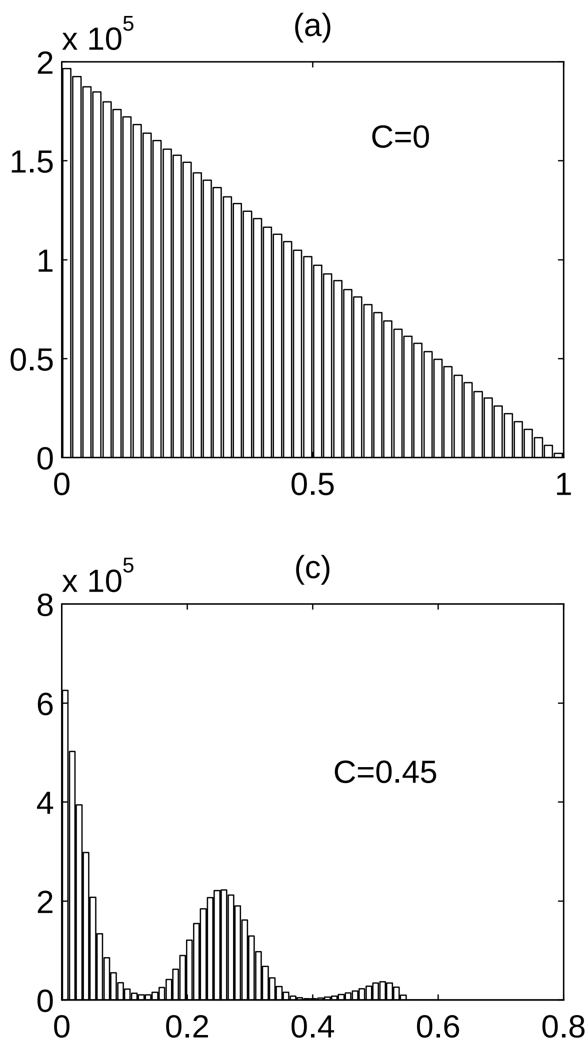
FIGURE CAPTIONS

Fig.1 - Distribution of the variable $|x_i - x_j|$ for several values of the coupling parameter $C = \frac{N}{N-1}c$ in a Bernoulli network of 100 units

Fig.2 - Distribution of the variable $x_i + x_{i+1} - 2x_{i+2}$ for several values of the coupling parameter $C = \frac{N}{N-1}c$ in a Bernoulli network of 100 units



(a)
x 10^5
2
1.5
1
0.5
0
0
0.5
1
C=0
(b)
x 10^5
2.5
2
1.5
1
0.5
0
0
0.5
1
C=0.2
(c)
x 10^5
8
6
4
2
0
0
0.2
0.4
0.6
0.8
C=0.45
(d)
x 10^6
5
4
3
2
1
0
0
0.5
1
C=0.6

Fig.1

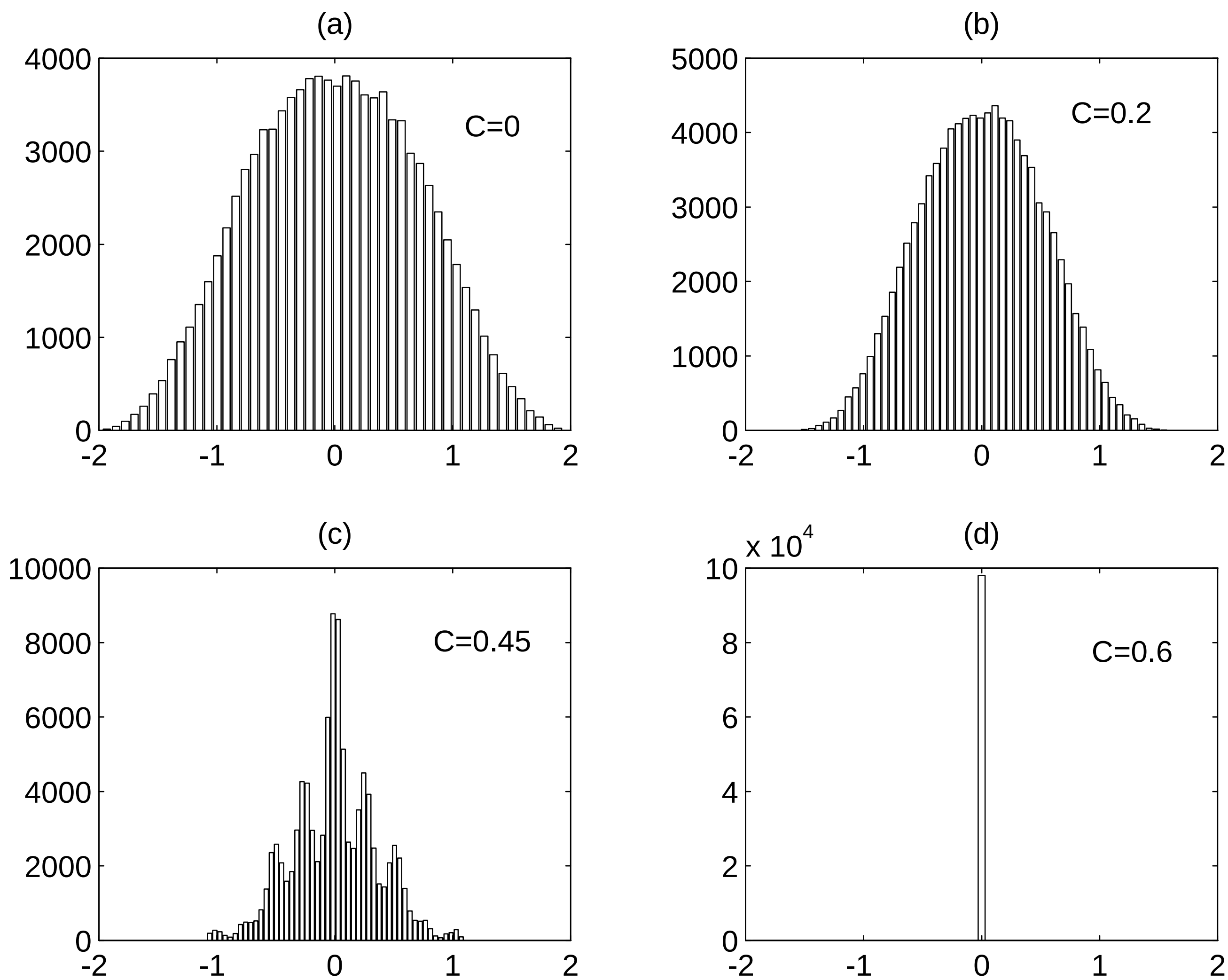

(a)
4000
3000
2000
1000
0
-2
-1
0
1
2
C=0
(b)
5000
4000
3000
2000
1000
0
-2
-1
0
1
2
C=0.2
(c)
10000
8000
6000
4000
2000
0
-2
-1
0
1
2
C=0.45
(d)
x 10^4
10
8
6
4
2
0
-2
-1
0
1
2
C=0.6

Fig.2